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The purity of impure public goods

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Abstract

In this paper, we provide a new perspective on the links between the analysis of the voluntary provision of pure and impure public goods. In particular, it is shown that the impure public good model can be transformed into a pure public good one. This innovative method not only leads to new comparative statics results, but also provides new insights on the impure public good model, for example, on causes of the nonneutrality of income transfers with regard to Nash equilibria in the impure public good case.

1 | INTRODUCTION

Voluntary ("private") provision of public goods has become a main topic in public economics with many empirical applications especially in the context of global public goods (see Buchholz & Sandler, 2021) and charitable donations (see Andreoni & Payne, 2013). While an agent's contributions to a *pure* public good (PPG) only generate benefits for the entire community, a contribution to an *impure* public good (IPG) in addition entails a private benefit for the contributing agent herself. In the real world examples for such IPGs are abundant reaching from military alliances (Murdoch & Sandler, 1984) to warm-glow-of-giving (Andreoni, 1989, 1990; Long, 2020). In the meantime, other applications as global environmental problems like climate change and biodiversity loss (Chan, 2019; Rübbelke, 2003), refugee protection (Betts, 2003), environmentally friendly consumption goods (Kotchen, 2005, 2006), and green electricity programs (Kotchen & Moore, 2007; Mitra & Moore, 2018) have attracted further attention to IPGs. In climate economics, private cobenefits or synonymously "ancillary benefits" have raised much interest both in empirical and theoretical research (see, e.g., Buchholz et al., 2020; Löschel et al., 2021). So the clean development mechanism (CDM) as part of the Kyoto Protocol had the dual objective of protecting the climate and supporting local sustainable development (SD). While the CDM's climate protection benefits are globally public, the local SD cobenefits are private from the point of view of a country hosting a CDM project. In a similar vein, the UN 2030 Agenda for Sustainable Development has formulated sustainable development goals (SDGs) that are targeted to global as well as country-specific benefits. This means, for instance, that those global climate change mitigation actions in one country that are taken in accordance with SDG 13 (climate action) are to be preferred when they simultaneously help reaching other SDGs, for example, by improving *local* access to drinking water.

The theoretical analysis of the IPG model, as for example, of the Nash equilibrium of the non-cooperative provision game, appears to be much more demanding than that of the PPG model (see Cornes & Sandler, 1984, 1994, 1996; Ribar & Wilhelm, 2002; Yildirim, 2014), simply because a third variable that reflects an agent's private cobenefits has to be included in the utility function (see Cornes & Sandler, 1996; pp. 255–272).[1] Moreover, technical restrictions of producing the public and the private characteristics of the IPG have to be taken into account, which adds further constraints to the agent's utility maximization thus leading to a points-rationing problem. Finally, standard analysis of the IPG model necessitates the construction of a virtual price system which makes the analysis rather involved (see Brumme et al., 2020; Cornes & Sandler, 1994; Kotchen, 2005; Rübbelke, 2002, 2003).

The PPG model can be considered as a special case of the IPG model as omission of the private characteristic generated by the IPG brings us back to the PPG case—which trivially is of no help for facilitating the analysis of the IPG model. In this paper, we will therefore develop a technique by which the IPG model is translated into a PPG one while still taking into account impure publicness, that is, the jointly produced private characteristics. This allows us to waive virtual price systems and to explore the IPG case within the framework of the well-understood PPG case. This novel approach has several advantages and helps us to gain new insights on the IPG model:

- It improves the understanding of the particularities of the impure case and helps elucidate the common features and differences between the IPG and the PPG model. For instance, in contrast to the PPG case an agent acting as a Nash player in the IPG model may always choose a strictly positive public good contribution irrespective of the size of the other agents' contributions. With our approach it also becomes clear from the outset that Warr neutrality as an important feature of the PPG model cannot be expected in the IPG case.
- It makes it possible to specify which properties of utility functions (as the primitive elements of the IPG model) entail essential properties of the Nash equilibrium, especially its existence and uniqueness and the partitioning of the entire set of agents into contributors and free-riders.
- It enables us to conduct in a straightforward way a wide range of comparative statics exercises for Nash equilibria. In contrast to the existing literature, we are not restricted to deal with the effects that parameter changes have on a single price-taking agent's demand for her IPG contributions and her private consumption, but with our approach we can also take the reactions of the other agents and thus the repercussions on the entire Nash equilibrium into account.

[1]Vicary (1997) investigates the provision of a public characteristic by means of two different goods simultaneously: a pure public and an impure public good. Vicary (2000) then examines the case where there is private consumption adversely affecting the consumption of a public characteristic while this public characteristic is produced by donations.

The remainder of our paper will be organized as follows. In Section 2, we convert the agents' preferences for an IPG model into preferences of a PPG for which, as a special feature and as a deviation from the standard PPG model, there may be satiation with respect to private consumption. In Section 3, we show how, based on this transformation, the Nash equilibrium of voluntary provision of an IPG can be determined in a simple way and contributors and free-riders can be identified. Properties of the underlying utility function on which this conversion depends then are discussed in Section 4, while based on the purity of IPGs a comparison between basic features of the IPG and the PPG model is made in Section 5. In Section 6 it is shown how comparative statics analyses can easily be conducted with the help of our approach and how their results depend on properties of the agents' utility functions. Section 7 concludes, summarizing the basic insights of our paper.

2 | TRANSFORMING IMPURE PUBLIC GOOD PREFERENCES INTO PURE PUBLIC GOOD ONES

Let n agents $i = 1, \ldots, n$ with IPG utility functions $U^i(c_i, Z_i, z_i) = U^i(c_i, \alpha_i G, \beta_i g_i)$ be given where c_i is agent i's private consumption, g_i is her public good contribution measured in units of the private good and G is the aggregate public good contribution made by all n agents. The parameters α_i and β_i indicate how agent i's contributions to the public good transform into her contribution's public and private characteristic Z_i and z_i (see Cornes & Sandler, 1984, 1994, 1996). To facilitate the exposition we will assume $\alpha_i = \beta_i = 1$ for all agents $i = 1, \ldots, n$ for the most part of the paper so that we can identify Z_i with G and z_i with g_i.

The utility function $U^i(c_i, G, g_i)$ of each agent i is defined for all $(c_i, G, g_i) \in \mathbb{R}^3_+$, it is twice partially differentiable (with continuous second derivatives), and strictly quasi-concave. For the first partial derivatives of $U^i(c_i, G, g_i)$ we assume $U^i_k > 0$ for $k = 1, 2, 3$, that is, that utility is increasing in all three variables. Having $U^i_3 > 0$ indicates the impurity of the public good, that is, that agent i derives positive private cobenefits from her public good contribution g_i and that these "ancillary benefits" are increasing in g_i. By w_i we denote agent i's income, which can either be spent on private consumption c_i or on her IPG contribution g_i.

Now define

$$\hat{c}_i(G) := \arg\max_{c_i} U^i(c_i, G, w_i - c_i), \tag{1}$$

which is agent i's optimal choice of private consumption c_i for the given IPG supply G and which—due to the quasi-concavity of $U^i(c_i, G, g_i)$—is uniquely determined (as can be observed from Figure A2a,b that are employed in the Proof of Lemma 1 in Appendix A).

For a detailed specification of $\hat{c}_i(G)$ let $\mu^i_G(c_i, g_i)$ be the marginal rate of substitution between agent i's private consumption c_i and her public good contribution g_i when public good supply is G, which indicates how many units of g_i agent i is willing to give up to get an additional unit of c_i. Formally, $\mu^i_G(c_i, g_i) = \frac{U^i_1(c_i, G, g_i)}{U^i_3(c_i, G, g_i)}$. In a c_i–g_i-diagram (see Figure 1) $\mu^i_G(c_i', g_i')$ is the absolute value of the slope of the indifference curve that results for the given G at some point (c_i', g_i').

For $\mu^i_G(c_i, g_i)$ we make the following assumption:

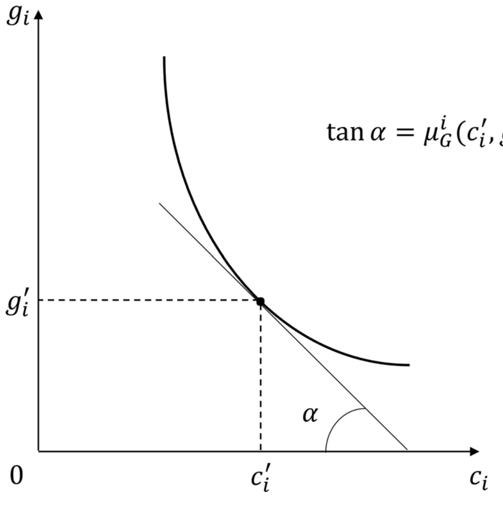

FIGURE 1 Visualization of assumption (A1)

(A1) $\lim_{c_i \to 0} \mu_G^i(c_i, g_i) > 1$ *for all* $G > 0$ *and all* $g_i > 0$.

Based on assumption (A1) and observing that $U_1^i(c_i, G, w_i - c_i) - U_3^i(c_i, G, w_i - c_i) = U_3^i(c_i, G, w_i - c_i)(\mu_G^i(c_i, w_i - c_i) - 1)$ two cases can be distinguished:

Lemma 1. *Assumption* (A1) *implies* $\hat{c}_i(G) > 0$ *for all* $G > 0$. *Moreover,*

(i) *if* $\lim_{g_i \to 0} \mu_G^i(w_i, g_i) \geq 1$ *holds for the given* G, *then* $\hat{c}_i(G) = w_i$.

(ii) *if* $\lim_{g_i \to 0} \mu_G^i(w_i, g_i) < 1$ *holds for the given* G, *then* $\hat{c}_i(G) < w_i$ *and* $\mu_G^i(\hat{c}_i(G), w_i - \hat{c}_i(G)) = 1$.

Proof. See Appendix A. □

In Case (i) of Lemma 1 it pays for agent i to increase her private consumption up to her initial endowment w_i. In Case (ii) of Lemma 1 the marginal private cobenefits of agent i's public good contribution start to outweigh her marginal benefits of private consumption at $\hat{c}_i(G)$ so that agent i does not want to increase private consumption beyond that level. In this sense, $\hat{c}_i(G)$ could be interpreted as agent i's *satiation level* for private consumption given public good supply G[2]. In the c_i–g_i-diagram this means that the indifference curve associated with G is tangential to the straight line $g_i = w_i - c_i$ that has slope -1.

[2]Without referring to IPGs from the start, we could as well consider a PPG model, which in contrast to the standard model allows for satiation of private consumption. Satiation of private consumption can be considered as a reasonable assumption, for example, in the context of global environmental problems, such as climate change: If environmental conditions become more and more hostile, agents and countries may no longer benefit from increased private consumption.

Example To visualize the determination of $\hat{c}_i(G)$ we consider an example in which some agent i's preferences are $U^i(c_i, G, g_i) = c_i^{\frac{1}{2}}\varphi_i(G) + (a_i + g_i)^{\frac{1}{2}}$ where $a_i \geq 0$ and $\varphi_i(G)$ is a strictly monotone and concave (and differentiable) function with $\varphi_i(0) = 0$. Then $\mu_G^i(c_i, g_i) = \frac{U_1^i(c_i, G, g_i)}{U_3^i(c_i, G, g_i)} = \left(\frac{a_i + g_i}{c_i}\right)^{\frac{1}{2}}\varphi(G)$, which implies that the condition in part (i) of Lemma 1 is fulfilled for the given G if $\varphi_i(G) \geq \left(\frac{w_i}{a_i}\right)^{\frac{1}{2}}$. If $\varphi_i(G) < \left(\frac{w_i}{a_i}\right)^{\frac{1}{2}}$ we are in Case (ii) instead. The condition $\mu_G^i(\hat{c}_i(G), w_i - \hat{c}_i(G)) = 1$ then implies $\hat{c}_i(G) = \frac{\varphi_i(G)^2(a_i + w_i)}{1 + \varphi_i(G)^2} < w_i$. As the function $\varphi_i(G)$ has been assumed to be strictly monotone increasing, Case (i) is more likely if G is large. If $a_i = 0$ or if $a_i > 0$ and $\varphi_i(G)$ is bounded above by $\left(\frac{w_i}{a_i}\right)^{\frac{1}{2}}$, we are in Case (ii) for any level of G.

Based on the distinction provided by (i) and (ii) we now consider changes of G and show how the original IPG utility function $U^i(c_i, G, g_i)$ with its three arguments can be transformed into an auxiliary PPG utility function $u^i(c_i, G)$ with only two arguments by letting

$$u^i(c_i, G) = \begin{cases} U^i(c_i, G, w_i - c_i) & \text{if} \quad c_i < \hat{c}_i(G) \\ U^i(\hat{c}_i(G), G, w_i - \hat{c}_i(G)) & \text{if} \quad c_i \geq \hat{c}_i(G) \end{cases} \tag{2}$$

The indifference curves $c_i^{\bar{u}_i}(G)$, along which agent i's utility $u^i(c_i, G)$ is constant, have the slope $\frac{\partial c_i^{\bar{u}_i}}{\partial G} = -\frac{U_1^i - U_3^i}{U_2^i}$ to the left of $\hat{c}_i(G)$. Since in this region $U_1^i - U_3^i > 0$ and $U_2^i > 0$ holds, the indifference curves are decreasing for $\hat{c}_i(G) < w_i$ and due to quasi-concavity, they are also convex there. For $c_i > \hat{c}_i(G)$ the indifference curve passing through $(\hat{c}_i(G), G)$ is a flat line to the right of it (see Figure 2). For a further description of the curve $\hat{c}_i(G)$, which connects these "kinks" of the indifference curves, we make an additional assumption on $\mu_G^i(c_i, g_i)$, which then gives Lemma 2:

(A2) *There exists a* $\underline{G}_i \in (0, \infty]$ *so that* $\lim_{g_i \to 0} \mu_G^i(w_i, g_i) < 1$ *holds for all* $G < \underline{G}_i$ *and* $\mu_G^i(c_i, g_i)$ *is increasing in* G.

Lemma 2. $\hat{c}_i(G) < w_i$ *holds for all* $G < \underline{G}_i$ *and in this region* $\hat{c}_i(G)$ *is increasing in* G.

Proof. See Appendix A. □

This Lemma says that $\hat{c}_i(G)$ is smaller than agent i's endowment w_i for small levels of G and does not decrease with public good supply. Concerning the shape of her $\hat{c}_i(G)$-curve agent i can be of either of two types.

Type 1: For agent i there exists a public good level $\hat{G}_i < \infty$, so that for all $G \geq \hat{G}_i$ we are in Case (i) as defined above. Then $\hat{c}_i(G) < w_i$ only occurs for small levels of G. At $\hat{G}_i$ the curve $\hat{c}_i(G)$ intersects the vertical line passing through agent i's endowment point $(w_i, 0)$, that is, $\hat{c}_i(G) = w_i$ holds (see Figure 2). In this case, we set $\hat{c}_i := w_i$.

Type 2: For all $G \geq 0$ we are in Case (ii) so that $\hat{c}_i(G) < w_i$ for all $G > 0$. In this case, we set $\hat{G}_i = \infty$. Since $\hat{c}_i(G)$ is increasing according to Lemma 2, $\hat{c}_i := \lim_{G \to \infty} \hat{c}_i(G) \leq w_i$ exists and $\hat{c}_i(G) < \hat{c}_i$ holds for all $G > 0$. For agents of type 2 we have two subcases, that is, agent i either

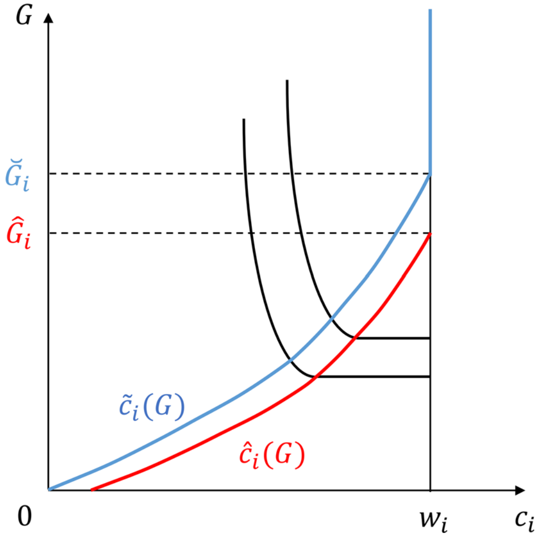

FIGURE 2 Indifference curves, the curve $\hat{c}_i(G)$ and the expansion path $\tilde{c}_i(G)$ for type 1 agents.

is of type 2a when $\hat{c}_i = w_i$ holds (see Figure 3a), or she is of type 2b when $\hat{c}_i < w_i$ holds (see Figure 3b).

Example continued. In the example considered above $\hat{c}_i(G) = \frac{\varphi_i(G)^2(a_i + w_i)}{1 + \varphi_i(G)^2}$ is an increasing and convex function for $G \in [0, \hat{G}_i] = [0, w_i]$ with $\hat{c}_i(0) = 0$ as depicted in the figures. Let $\bar{\varphi}_i := \lim_{G \to 0} \varphi_i(G)$. If $a_i > 1$ and $\bar{\varphi}_i > \left(\frac{w_i}{a_i}\right)^{\frac{1}{2}}$, agent i is of type 1. Then $\hat{c}_i(G)$ intersects the vertical line through $(w_i, 0)$ at $\hat{G}_i < \infty$ (see Figure 2), which is obtained from $\hat{c}_i(\hat{G}_i) = w_i$ as $\hat{G}_i = \varphi_i^{-1}\left(\left(\frac{w_i}{a_i}\right)^{\frac{1}{2}}\right) < \infty$. In all other cases agent i is of type 2 with $\hat{c}_i = \lim_{G \to \infty} \hat{c}_i(G) = \lim_{G \to \infty} \frac{a_i + w_i}{1 + \frac{1}{\varphi_i(G)^2}} = \frac{a_i + w_i}{1 + \frac{1}{\bar{\varphi}_i}}$. If $a_i = 0$ we have $\hat{c}_i = w_i$ if $\bar{\varphi}_i = \infty$, and agent i is of type 2a as depicted in Figure 3a. If instead $\bar{\varphi}_i < \infty$ she is of type 2b with $\hat{c}_i = \frac{w_i}{1 + \frac{1}{\bar{\varphi}_i}} < w_i$ as depicted in Figure 3b. In the remaining cases with $a_i > 1$ type 2a is obtained if $\bar{\varphi}_i = \left(\frac{w_i}{a_i}\right)^{\frac{1}{2}}$ and type 2b if $\bar{\varphi}_i < \left(\frac{w_i}{a_i}\right)^{\frac{1}{2}}$.

3 | DETERMINING THE NASH EQUILIBRIUM IN THE IMPURE PUBLIC GOOD MODEL

We now assume that the public good G is, as in the standard PPG case, produced by a summation technology for which the marginal rate of transformation between the private and the public good is equal to one for all agents $i = 1, ..., n$. In this setting we consider the voluntary provision game in which each agent behaves according to the Nash hypothesis, that

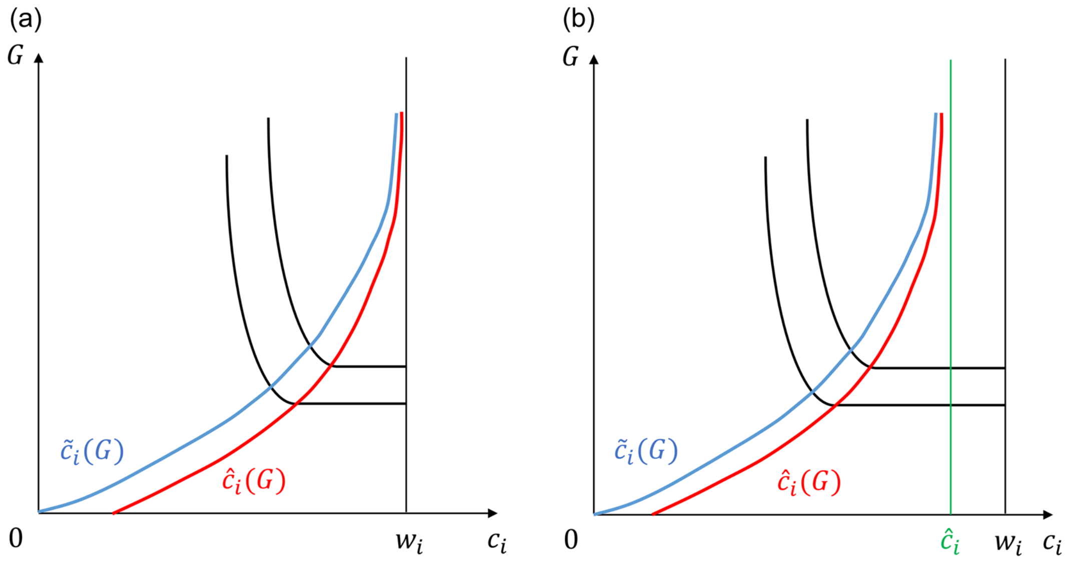

FIGURE 3 (a) Indifference curves, the curve $\hat{c}_i(G)$ and the expansion path $\tilde{c}_i(G)$ for type 2a agents. (b) Indifference curves, the curve $\hat{c}_i(G)$ and the expansion path $\tilde{c}_i(G)$ for type 2b agents.

is, she chooses her public good contribution given the public good contributions of all other agents.

The characterization of the Nash equilibria of this game will be based upon the *expansion paths* $\tilde{c}_i(G)$ along which the marginal rate of substitution $m^i(c_i, G)$ between the private and the public good coincides with the marginal rate of transformation that is equal to one, that is, $m^i(c_i, G) := \frac{u_1^i(c_i, G)}{u_2^i(c_i, G)} = 1$ holds (where $u_1^i(c_i, G)$ and $u_2^i(c_i, G)$ are the first partial derivatives of the auxiliary utility function $u^i(c_i, G)$). (Note that we are now no longer in a c_i–g_i-diagram but in a c_i–G-diagram so that the marginal rate of substitution $m^i(c_i, G)$ indicates how many units of G agent i is willing to give up to get an additional unit of c_i).

Under our assumptions the function $m^i(c_i, G)$ clearly is differentiable. Denoting the first partial derivatives of $m^i(c_i, G)$ by $m_1^i(c_i, G)$ and $m_2^i(c_i, G)$, respectively, we then make the following assumption:

(A3) *$m_1^i(c_i, G) < 0$ and $m_2^i(c_i, G) > 0$ holds at all points (c_i, G) at which $m^i(c_i, G) > 0$. Moreover, we have $\lim_{c_i \to 0} m^i(c_i, G) > 1$ for each $G > 0$ and $\lim_{G \to \infty} m^i(c_i, G) > 1$ for all $c_i < \hat{c}_i$.*

The first part of this assumption means that in a c_i–G-diagram the indifference curves become steeper (flatter) when one moves vertically upwards (horizontally to the left). This assumption is visualized in Figure A1 in Appendix A.

Given (A3) important properties of expansion paths are presented by the following Lemma.

Lemma 3. *If preferences of agent i satisfy* (A3), *then for all $G \leq \hat{G}_i$*

(i) *there exists a unique $\tilde{c}_i(G) < w_i$ where $m^i(\tilde{c}_i(G), G) = 1$ holds,*

(ii) *the expansion path* $\tilde{c}_i(G)$ *lies to the left of the curve* $\hat{c}_i(G)$ *and is increasing in* G.

Proof. See Appendix A. □

Based on Lemma 3 it is now possible to describe how the expansion paths of type 1 and type 2 agents differ.

Proposition 1. *Let* $(A3)$ *be fulfilled.*

(i) *If agent* i *is of type* 1, *there is a* $\breve{G}_i < \infty$ *with* $\breve{G}_i > \hat{G}_i$ *for which* $\tilde{c}_i(\breve{G}_i) = w_i$ *holds.*

(ii) *If agent* i *is of type* 2, *then* $\lim_{G \to \infty} \tilde{c}_i(G) = \hat{c}_i$ *and* $\breve{G}_i = \infty$.

Proof. See Appendix A. □

For an agent i of type 1 $(w_i, \breve{G}_i)$ is the point at which the expansion path $\tilde{c}_i(G)$ intersects the vertical line through the endowment point $(w_i, 0)$. In this case, we continue the expansion path by letting $\tilde{c}_i(G) = w_i$ for all $G > \breve{G}_i$. For an agent of type 2 the expansion path $\tilde{c}_i(G)$ instead asymptotically approaches the vertical straight line through $(\hat{c}_i, 0)$. For a visualization Figures 2 and 3a,b depict the paths $\hat{c}_i(G)$ and $\tilde{c}_i(G)$ in red and blue for the different agents' types 1, 2a, and 2b, respectively. Indifference curves are marked in black.

As along $\tilde{c}_i(G)$ agent i's marginal rate of substitution between the private and the public good coincides with the marginal rate of transformation in the range $G < \breve{G}_i$, the expansion path $\tilde{c}_i(G)$ describes the positions a contributing agent i will attain in a Nash equilibrium: A small deviation from such a position, by either increasing or decreasing her public good contribution, would make her worse off. If instead $G \geq \breve{G}_i$, agent i's marginal rate of substitution between the private and the public good exceeds the marginal rate of transformation in (w_i, G) so that in this case agent i attains a Nash equilibrium position as a noncontributing free-rider. This, in particular, means that agent i stops contributing to the public good if the other agents in total contribute at least $\breve{G}_i$. In this sense $\breve{G}_i$ can be interpreted as agent i's "dropout level."

Following the basic ideas underlying the *aggregative game approach* (see, e.g., Cornes & Hartley, 2007), public good supply G^N in the Nash equilibrium can now be characterized—not only for PPGs but, after the conversion of their utility functions, also for IPGs—by the following Proposition, which is based on the resource requirement function $\Phi(G) := G + \sum_{i=1}^{n} \tilde{c}_i(G)$.

Proposition 2. *Given* $(A3)$, *there is a unique Nash equilibrium* $(c_1^N, ..., c_n^N, G^N)$ *where public good supply* G^N *is given by*

$$\Phi(G^N) = \sum_{i=1}^{n} w_i, \tag{3}$$

while private good consumption of agent $i = 1, ..., n$ *is* $c_i^N = \tilde{c}_i(G^N)$.

Proof. In a Nash equilibrium the aggregate budget constraint must be fulfilled and each agent i must be in a Nash equilibrium position on $\tilde{c}_i(G)$. This means that in a Nash equilibrium condition (3) must hold. Given (A3) a public good level G^N that satisfies this condition exists and—as well as the corresponding private consumption levels $c_i^N = \tilde{c}_i(G^N)$—is uniquely determined. This directly follows by applying the intermediate value theorem to the function $\Phi(G)$, which is continuous and strictly monotonically increasing in G, since on the one hand we have $\Phi(0) \le \sum_{i=1}^n \lim_{G\to 0} \tilde{c}_i(G) < \sum_{i=1}^n \lim_{G\to 0} \hat{c}_i(G) < \sum_{i=1}^n w_i$, which is a consequence of assumption (A2) and $\tilde{c}_i(G) < \hat{c}_i(G)$ for $G < \hat{G}_i$ according to Lemma 3, and on the other hand $\lim_{G\to\infty} \Phi(G) \ge \lim_{G\to\infty} G = \infty$ holds as $\tilde{c}_i(G)$ clearly cannot become negative. If $\tilde{c}_i(G^N) < w_i$, agent i contributes $g_i^N = w_i - \tilde{c}_i(G^N) > 0$ to the public good while an agent i contributes nothing if $\tilde{c}_i(G^N) = w_i$. □

Another proof of the existence and uniqueness of the Nash equilibrium in the IPG economy has been provided by Kotchen (2007). There a variant of the aggregative game approach was used that, as in Cornes and Hartley (2007), was based on "replacement functions" and in which normality was introduced by a direct assumption on best-response curves. Our approach is based on properties of expansion paths instead, which in microeconomic theory are usually used to characterize the normality of preferences. In this way, it also becomes easily possible to trace the normality assumption back to the properties of the IPG utility function. How this can be done will be expounded in the subsequent section.

4 | PROPERTIES OF IMPURE PUBLIC GOOD PREFERENCES THAT ALLOW THE CONVERSION

We now show which properties of the originally given IPG utility function $U^i(c_i, G, g_i)$ as the primitives of the model ensure that assumption (A3) is satisfied.[3]

Proposition 3. *For a given IPG utility function $U^i(c_i, G, g_i)$ assumption (A3) is satisfied if and only if $U_{11}^i + U_{33}^i - 2U_{13}^i - m^i(U_{12}^i - U_{23}^i) < 0$ and $U_{12}^i - U_{23}^i - m^i U_{22}^i > 0$.*

Proof. See Appendix A. □

As a direct consequence of Proposition 3 a sufficient condition for (A3) is provided by the following assumption.

(A4) *For the given IPG utility function U^i besides $U_{kk}^i < 0$ for $k = 1, 2, 3$ we have $U_{12}^i = U_{21}^i \ge 0$, $U_{13}^i = U_{31}^i \ge 0$, and $U_{23}^i = U_{32}^i \le 0$.*

The signs of the cross derivatives of $U^i(c_i, G, g_i)$ that underlie (A4) can be motivated as follows:

(i) $U_{12}^i = U_{21}^i \ge 0$ means that a higher level of public good supply (e.g., a better infrastructure or an improvement of environmental quality) increases marginal utility of private

[3] To satisfy the required limit properties of $m^i(c_i, G)$ is technically more demanding, but not difficult to achieve: Looking at the numerator and denominator of $m^i = \frac{U_1 - U_3}{U_2}$ we get $\lim_{G\to\infty} m^i(c_i, G) = \infty$ if $\lim_{G\to\infty} U_2^i = 0$ and (A4) holds. $\lim_{c_i\to 0} m^i(c_i, G) = 0$ is shown by an analogous argument.

consumption. This is an appealing assumption as, for example, the quality of living is enhanced through better public transport facilities and less air pollution.

(ii) $U^i_{13} = U^i_{31} \geq 0$ means that there is complementarity between utility from private consumption and the cobenefits of the public good contributions, which for example, in the warm-glow-of-giving version of the IPG model means that higher private consumption, that is, improved material well-being, increases an agent's marginal joy from giving. This is also a reasonable assumption.

(iii) $U^i_{23} = U^i_{32} \leq 0$ means that a higher supply of the public good will not increase the individual cobenefits. In the case of warm-glow-of-giving this can be explained by an agent's desire to stand out from the others in her contribution to the public good (see also Yildirim, 2014, p. 102). The more the others already contribute, the less the agent appreciates her own contribution because she can no longer consider herself as a moral pioneer. An alternative interpretation may be that after trespassing some threshold an agent's contribution becomes less urgent for providing enough of the public good that is, for example, required for avoiding the danger of a climate catastrophe.

Assumption (A4) is clearly fulfilled if the cobenefit component enters the IPG utility function in an additive way, that is, if we have $U^i(c_i, G, g_i) = v^i(c_i, G) + \chi_i(g_i)$ as in the example used in Section 2, where $v^i(c_i, G)$ is a PPG utility function with $v^i_{11} < 0$, $v^i_{22} < 0$ and $v^i_{12} \geq 0$ and $\chi_i(g_i)$ is monotonically increasing and concave. This especially holds if the IPG utility function is additively separable, that is, if $U^i(c_i, G, g_i) = \phi_i(c_i) + \varphi_i(G) + \chi_i(g_i)$ with monotone increasing and concave functions $\phi_i(c_i)$, $\varphi_i(G)$ and $\chi_i(g_i)$ for which $\phi_i'(0) > \chi_i'(w_i)$. If $U^i(c_i, G, g_i) = \phi_i(c_i) + h^i(G, g_i)$ we need $U^i_{23} = h^i_{12} \leq 0$ for getting (A4).[4]

For additively separable utility functions the path $\hat{c}_i(G)$ has a very specific shape as in this case we have $U^i(c_i, G, w_i - c_i) = \phi_i(c_i) - \chi_i(w_i - c_i)$, which is independent of public good supply G. Hence, we have the following result:

Proposition 4. *If the utility function is additively separable,* $\hat{c}_i(G) = \hat{c}_i$ *is constant for all* G.

Applying the reasoning from Section 2, we see that for additively separable utility functions agent i is of type 1 with $\hat{c}_i = w_i$ if $U^i_1(w_i, G, 0) = \phi_i'(w_i) \geq \chi_i'(0) = U^i_3(w_i, G, 0)$. If instead $\phi_i'(w_i) < \chi_i'(0)$, agent i is of type 2 and we have $\hat{c}_i < w_i$. In this case $\hat{c}_i$ is characterized by the first-order condition $\phi_i'(\hat{c}_i) = \chi_i'(w_i - \hat{c}_i)$. This generalizes the example presented in Section 2.

5 | A COMPARISON BETWEEN THE IMPURE AND THE PPG MODEL

The approach presented above sheds some new light on the IPG model and moreover allows us to obtain new results for this model in a simple way. Interpreting the IPG model as a variant of the PPG model also leads to a better understanding of the differences that exist between the two types of public good models.

[4]When especially $h^i(G, g_i) = h^i(\alpha_i G + \beta_i g_i)$, this holds if the function h^i is concave, which provides a generalization of the utility functions used by Yildirim (2014; p. 103) where $\phi_i(c_i) = \ln c_i$ and $h^i(G, g_i) = \ln(\alpha_i G + \beta_i g_i)$.

- From a methodological viewpoint, our approach allows a uniform analytical treatment of voluntary public good provision for PPGs and IPGs. Based on the equilibrium condition (3) this particularly holds for the proofs of existence and uniqueness of the Nash equilibrium. Hence the analysis of Nash equilibria in the IPG model does not require specific and demanding methods—and results that have been obtained for the PPG model can be directly transferred to the IPG model.
- Concerning the question which agents are contributors or free-riders it turns out that virtually there is no difference between the IPG and the PPG model when all agents are of type 1: Each agent i stops contributing to the IPG and becomes a free-rider if the other agents' public good supply exceeds her dropout level $\breve{G}_i$, which is finite in this case. If the economy is enlarged by adding the same agents of type 1, public good supply in the Nash equilibrium then is bounded from above by the largest dropout level—just as in the case of a PPG (see Andreoni, 1988; Chamberlin, 1974; McGuire, 1974). If, however, some agents are of type 2, they will always contribute to the public good—irrespective of how high the contributions of the other agents are. This marks a sharp contrast of the IPG model to the standard PPG model.

To explain the second point in more detail we consider Nash equilibria in an IPG economy that consist of an arbitrary number k of the same agent of type 2. This agent's income is w and her preferences are assumed to satisfy (A3) so that her expansion path $\tilde{c}(G)$ is increasing with $\hat{c} = \lim_{G\to\infty} \tilde{c}(G)$. Depending on k IPG supply in the Nash equilibrium is denoted by $G^N(k)$ and each agent's public good contribution by $g^N(k)$. Then we have the following result, which—by distinguishing agents of type 2a and type 2b—provides a variation and refinement of a result by Yildirim (2014).

Proposition 5. *Let the number k of type 2 agents go to infinity. Then we have $\lim_{k\to\infty} G^N(k) = \infty$ and $\lim_{k\to\infty} g^N(k) = \hat{g} := w - \hat{c}$. If the agent is of type 2a, we have $\hat{g} = 0$ while for an agent of type 2b $\hat{g} > 0$.*

Proof. See Appendix A. □

It follows from this proposition that with an increasing number of agents IPG supply in the Nash equilibrium may go to infinity even though the individual public good contributions of all agents converge to zero.

From these observations, it also becomes obvious that the determination of the set of contributors in the Nash equilibrium for an IPG does not require any special algorithm. All agents of type 2 are contributors anyway, while for the agents of type 1 precisely the same methods for finding out contributors as in the case of a PPG (as that of Andreoni & McGuire, 1993) can also be applied to IPGs without any modification.

If (A4) holds also for IPGs (as for PPGs), a crowding-out effect exists, which means that if a single agent or a group of agents unilaterally increase their public good contributions, the other agents—by moving outward their increasing expansion paths—will react by increasing their private consumption, thus reducing their public good contributions. In the IPG case this crowding-out effect is strongly reduced when the outsiders are of type 2b and thus—as Nash players—always make a positive IPG contribution that is bounded away from zero. This is empirically important as unilateral action by a "climate club" in the sense of Nordhaus (2015) will be less confronted with countervailing behavior by the countries outside the club when the outsiders have large country-specific cobenefits.

Yet, the conditions that underlie (A4) and thus render the public good contributions of different agents to be strategic substitutes need not always hold for IPGs. In particular, it may happen that, while $U_{22}^i < 0$ and $U_{12}^i > 0$ are kept, in contrast to (A4) $U_{32}^i = U_{23}^i$ is positive and even is so large that $U_{32}^i = U_{23}^i > U_{12}^i - U_{22}^i$ holds. Then the conditions stated in Proposition 3 are not satisfied and an increased IPG supply enlarges the private cobenefits of agent i. For example, in the warm-glow-of-giving scenario, this means that the individual joy of giving is enhanced if a fundraising action is overall successful, and a bandwagon effect is triggered. Then the public good contributions of different agents are strategic complements so that a crowding-in effect results, that is, the increase of public good contributions by other agents motivates an agent to increase her own contributions too, and leadership in this sense works as desired (see Buchholz & Sandler, 2017; p. 597, for a further discussion).

6 | COMPARATIVE STATICS

Through the conversion of the IPG model a broad variety of comparative statics exercises for the IPG model can easily be conducted. In this context note that in previous studies for the standard IPG model (see Cornes & Sandler, 1994, 1996; Kotchen, 2005; Rübbelke, 2002) the comparative statics analysis had mostly been restricted to the examination of demand reactions of a single agent without considering the Nash equilibrium as such.[5] With our approach, however, it instead becomes easily possible to provide an explicit analysis of the changes of the entire Nash equilibrium also in the standard IPG framework when some exogenous factors vary. In this context a key role is played by shifts of expansion paths that are caused by parameter changes. This will be shown by Proposition 6 which provides a very general comparative statics effect upon which most of the subsequent special (and largely new) comparative statics results will rest.

6.1 | Changes of expansion paths

As a first step we assume that initial endowments $w_1, ..., w_n$ are fixed, but that some agent j's expansion path is changed exogenously through some parameter change.

Proposition 6. *Given (A4), a shift of some agent j's expansion path $\tilde{c}_j(G)$ to the left (right) leads to an increase (a decrease) of IPG supply G^N and of private consumption and utility of all agents $i \neq j$ in the Nash equilibrium.*

[5]Some comparative statics analyses for the Nash equilibrium in the standard PPG case can be found in Bergstrom et al. (1986) and Cornes and Hartley (2007). In Cornes and Hartley (2007), where use of the aggregative game approach is made, the IPG case is only briefly addressed in the concluding section without an explicit theoretical treatment. For a specific version of the IPG model which, however, is quite different from the classical IPG model as treated in this paper, Kotchen (2006) provides some comparative statics results for the Nash equilibrium, while some comparative statics results for the cooperative Lindahl equilibrium in the standard IPG model are presented by Chan and Dinelli (2020).

Proof. Looking at condition (3), that is, at $G^N + \sum_{i=1}^{n} \tilde{c}_i(G^N) = \sum_{i=1}^{n} w_i$, which characterizes public good supply G^N in the Nash equilibrium, we see that at G^N the left-hand side of (3) is decreased when some agent j's expansion path shifts to the left, while the right-hand side does not change. As (A4) implies that all expansion paths are increasing, restoring equality of both sides of (3) requires that public good supply increases and all agents except agent j move outwards their invariant expansion paths. If some agent j's expansion path is shifted to the right, these results are clearly reversed. □

A shift of agent j's expansion path to the left can, for example, be caused by an increase of this agent's preferences for the public good. This is represented by an increase of her marginal willingness to pay for the public good in any point (c_j, G), which is the reciprocal of the marginal rate of substitution between the private and the public good.

Proposition 7. *Given* (*A*4), *an increase of some agent j's preferences for the IPG shifts this agent's expansion path closer to the G-axis and thus leads to the effects on the Nash equilibrium as described by Proposition* 6.

Proof. The assertion follows from convexity of agent j's indifference curves, which is implied by (A4). □

In the IPG case, preferences for the public good not only—as in the case of a PPG—can increase through a decrease of the marginal utility of private consumption U_1^j or through an increase of the marginal utility of the public good U_2^j, but also through an increase of U_3^j, that is, by a higher marginal utility of the private cobenefits of agent j's public good contribution. In this way private consumption becomes less attractive for agent j, thus raising her preference for the public good.

6.2 | Income changes

In the case of an IPG—in stark contrast to the case of a PPG—the change of some agent j's endowment w_j not only makes the economy richer, but also induces a decrease of agent j's marginal willingness to pay for the public good at any point (c_j, G) — even though agent j's preferences as given by the utility function $U^j(c_j, G, g_j)$ remain the same.

Proposition 8. *If agent j's preferences satisfy* (*A*4), *then an increase of agent j's endowment w_j leads to a lower marginal willingness to pay for the IPG and thus to a shift of the expansion path to the right.*

Proof. See Appendix A. □

The result in Proposition 8 may appear a little surprising or even "strange." One might rather expect that increasing the income of an agent would increase her interest in the public good. The main reason for this counterintuitive result that can easily be explained in our approach is that at any point (c_j, G) after an increase of w_j agent j's public good contributions $w_j - c_j$ increase so that due to $U^j_{33} < 0$ her marginal private cobenefits $U^j_3(c_j, G, w_j - c_j)$ become smaller.

This explanation can be illustrated for the case of an additively separable utility function $U^j(c_j, G, g_j) = \phi_j(c_j) + \varphi_j(G) + \psi_j(g_j)$ for which $m^j(c_j, G) = \frac{\phi_j'(c_j) - \psi_j'(w_j - c_j)}{\varphi_j'(G)}$. As $\psi_j''(g_j) < 0$, an increase of w_j and thus of $g_j = w_j - c_j$ increases the numerator of this expression. Letting c_j and G be fixed, the marginal rate of substitution $m^j(c_j, G)$ then increases so that the marginal willingness to pay for the public good $\frac{1}{m^j(c_j, G)}$ at any (c_j, G) decreases.

The effect of some agent j's income increase on the Nash equilibrium clearly not only depends on this change of her willingness to pay for the IPG and the ensuing shift of the expansion path,[6] which, as described just now, tends to decrease public good supply. As an opposing effect it also has to be taken into account that an increase of w_j increases total income in the economy, that is, the right-hand side of the equilibrium condition (3) in Proposition 2. The next Proposition shows that, given assumption (A4), the second effect dominates the first one.

Proposition 9. *If agents' preferences satisfy (A4), then an increase of agent j's endowment w_j leads to an increase of public good supply, private consumption, and utility of all agents in the Nash equilibrium.*

Proof. See Appendix A. □

The effect on the Nash equilibrium that is caused by an increase of total income vanishes if we consider the effects of a *redistribution* of income between agents. For this case, a comparative statics result is provided by the next Proposition, which deals with a quite simple situation where the donor does not obtain private cobenefits from her public good contribution.

Proposition 10. *If (A4) is satisfied, then an income transfer from some agent k that has no private cobenefits to another agent j that has positive cobenefits leads to a decrease of public good supply and private consumption of all agents $i \neq j$ in the Nash equilibrium.*

Proof. The assertion is a direct implication of Proposition 6. □

[6]That an income increase changes the expansion path also implies that the standard concept of normality no longer applies to IPGs. This means that the changes of individual demand for an IPG (without considering reactions of other agents) cannot be described by movements along a fixed income expansion path as in the standard microeconomic consumer demand model (and the PPG case). To determine the changes of some agent j's individual demand for an IPG one has to consider the derivative of the corresponding first-order condition $\frac{U^j_1(w_j - G, G, G) - U^j_3(w_j - G, G, G)}{U^j_2(w_j - G, G, G)} = 1$ with respect to w_j, which gives $\frac{dG}{dw_j} = \frac{U^j_{11} - U^j_{12} - U^j_{13}}{U^j_{11} + U^j_{22} + U^j_{33} - 2(U^j_{12} + U^j_{13} - U^j_{23})}$. This expression is positive if assumption (A4) is fulfilled. Thus under this assumption normality in the sense that an increasing income leads to a higher individual demand also holds for an IPG.

If the donor has positive cobenefits too which, however, are smaller than those of the recipient, an analogous result holds (see Andreoni, 1989, 1990).

Against this background we also get a novel interpretation as to why Warr neutrality cannot be expected in the IPG case: Warr neutrality, which is a central result in public good theory, means that an income redistribution among the contributing agents that leaves the set of contributors unchanged will neither affect the level of public good supply nor the agents' private consumption levels in the Nash equilibrium (see Bergstrom et al., 1986; Cornes & Sandler, 1994, pp. 418–420, 1996; Faias et al., 2020; Warr, 1983). This neutrality holds for the standard PPG models as the expansion paths and so the equilibrium condition (3) are not changed by an income redistribution. For an IPG, however, the auxiliary preferences $u^j(c_j, G) = U^j(c_j, G, w_j - c_j)$ of an agent j and thus her expansion paths are *not* independent of her initial private good endowment w_j. As we can see from the proof of Proposition 9, an agent's expansion path in the case of an IPG will only stay the same after a change of her income if $U^j_{13} - U^j_{23} - U^j_{33} = 0$, that is, in particular $U^j_{13} = U^j_{23} = U^j_{33} = 0$ holds. This brings us either back to the case of a PPG—or leads us to a quasi-linear IPG utility function with linearity in its private cobenefit part.

6.3 | Changes of the productivity parameters α_j and β_j

Returning to the general version of the IPG model and complementing the analysis of Cornes and Sandler (1994, 1996) we can also analyze the shift of expansion paths caused by changes of the productivity parameters α_j and β_j. These parameters indicate how aggregate public good expenses G and agent j's own public good expenses $g_j = w_j - c_j$ are transformed into this agent's corresponding variables in characteristics space.

Proposition 11. *Let (A4) be given. Then public good supply G^N and private consumption and utility of all agents $i \neq j$ in the Nash equilibrium decrease if*

(i) α_j *increases and* $-\frac{U^j_{22}G}{U^j_2} > 1$ *holds,*

(ii) β_j *increases and* $-\frac{U^j_{33}G}{U^j_3} > 1$ *holds.*

Proof. See Appendix A. □

When agent j has additively separable preferences, only $U_2 + U_{22}G = \varphi_j' + \varphi_j''G$ determines the numerator of (i) in Proposition 11. Hence $-\frac{U^j_{22}G}{U^j_2} = -\frac{\varphi_j''G}{\varphi_j'} < 1$ in this case not only provides a sufficient, but also a necessary condition for an increase of public good supply (and the ensuing effects on private consumption and utility). If $-\frac{\varphi_j''G}{\varphi_j'} > 1$ a marginal increase of α_j thus induces the opposite effects.

7 | CONCLUSION

This paper is based on the insight that the IPG model can be traced back to a PPG model. By means of this transformation, which is possible for a broad class of IPG utility functions, we can show that many results on the non-cooperative Nash equilibrium[7] directly carry over from the pure to the impure case. So do the proofs for the existence and uniqueness of the Nash equilibrium, the methods for determining contributors and free-riders and comparative statics that rest upon shifts of the expansion paths.

Yet despite these similarities between the PPG and the IPG model, there are also considerable differences, which also become obvious through the lens of our approach. So in the impure case, there need not exist finite dropout levels for the public good when agents are of a specific type, which implies that such agents, acting as Nash players, always contribute—irrespective of the level of the other agents' contributions. Moreover, in contrast to the PPG model, the agents' expansion paths depend on their initial endowment. In particular, making an agent richer *reduces* her willingness to pay for the IPG, which at first sight seems paradoxical. The dependency of the expansion paths on income levels also explains why Warr neutrality, which is a central phenomenon for the private provision of a PPG, cannot be expected for an IPG from the start.

From the empirical perspective, our analysis of impure public goods is of high importance for research on international development and climate policy as current approaches regularly aim to "kill two birds with one stone," for example, seeking to attain different Sustainable Development Goals (SDGs) at the same time. Yet, as Tinbergen (1952) has already pointed out a long time ago, conflicts between different goals may arise and inefficiency is pending when instruments are overloaded with the ambition to reach more than one policy goal. In this context, this paper has shown that an agent's marginal willingness to pay for an IPG and thus her incentive to contribute to the IPG is reduced when she receives an income transfer. This may lead to a rather unexpected conflict between improved provision of an IPG on the one hand and distributional equity on the other.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

[7]This paper has dealt with voluntary provision of IPGs. Improving efficiency of allocation and attaining Pareto optimal outcomes poses particular challenges in the case of IPGs, which require elaborate solutions going beyond those in the PPG case. See Chan (2019) and Chan and Dinelli (2020) on this. In these papers, specific burden-sharing mechanisms are devised for attaining efficient solutions that partly make use of "multilateral financial mechanisms" (which combine burden-sharing with grants to the different countries through which contributions to a global IPG are financed) and also analyze the effects of income transfers.

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APPENDIX A

Visualization of Assumption (A3): In Figure A1 the horizontal (vertical) move, which makes the indifference curve flatter (steeper), is indicated by the move from point P' to point P_1' (P_2').

Proof of Lemma 1. Assumption (A1) implies that for all G the indifference curve through some $(c_i', w_i - c_i')$ is steeper than the straight line $g_i = w_i - c_i$ with slope -1 if c_i' is sufficiently close to zero (see Figure A2a,b). This gives $\hat{c}_i(G) > 0$ in both cases of Lemma 1. Given the assumption in Case (i) of Lemma 1 the indifference curve in each point of the budget line is steeper than the budget line so that we obtain $\hat{c}_i(G) = w_i$ (see Figure A2a). If, however, we are in Case (ii), there is a $c_i'' < w_i$ so that the indifference curve through $(c_i'', w_i - c_i'')$ is flatter there than the straight line $g_i = w_i - c_i$ and thus $\hat{c}_i(G) < c_i'' < w_i$ follows. At the point $\hat{P} = (\hat{c}_i(G), w_i - \hat{c}_i(G))$ the straight line $g_i = w_i - c_i$ then is tangential to an indifference curve, which gives $\mu_G^i(\hat{c}_i(G), w_i - \hat{c}_i(G)) = 1$ (see Figure A2b). □

Proof of Lemma 2. The first part of Lemma 2 directly follows from the second part of Lemma 1. Concerning the second part, assume that public good supply is increased from $G' < \underline{G}_i$ to some $G'' < \underline{G}_i$.

Assumption (A2) then implies that $\mu_{G''}^i(\hat{c}_i(G'), w_i - \hat{c}_i(G')) > \mu_{G'}^i(\hat{c}_i(G'), w_i - \hat{c}_i(G')) = 1$. In Figure A3 the indifference curve associated with G'' that passes through $P' = (\hat{c}_i(G'), w_i - \hat{c}_i(G'))$ (and which is marked in red) then becomes steeper there than the indifference curve associated with G', which is tangential to the budget line in P'. Hence $\hat{c}_i(G'') > \hat{c}_i(G')$. □

Proof of Lemma 3. (i) Since on the one hand $\lim_{c_i \to 0} m^i(c_i, G) > 1$ holds by assumption (A3) and on the other hand we have (by definition of $\hat{c}_i(G)$) $m^i(\hat{c}_i(G), G) = 0$ for $G \leq \hat{G}_i$

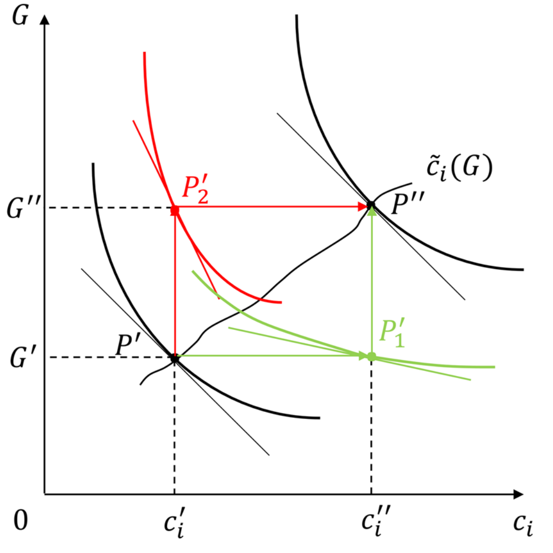

FIGURE A1 Visualization of assumption (A3)

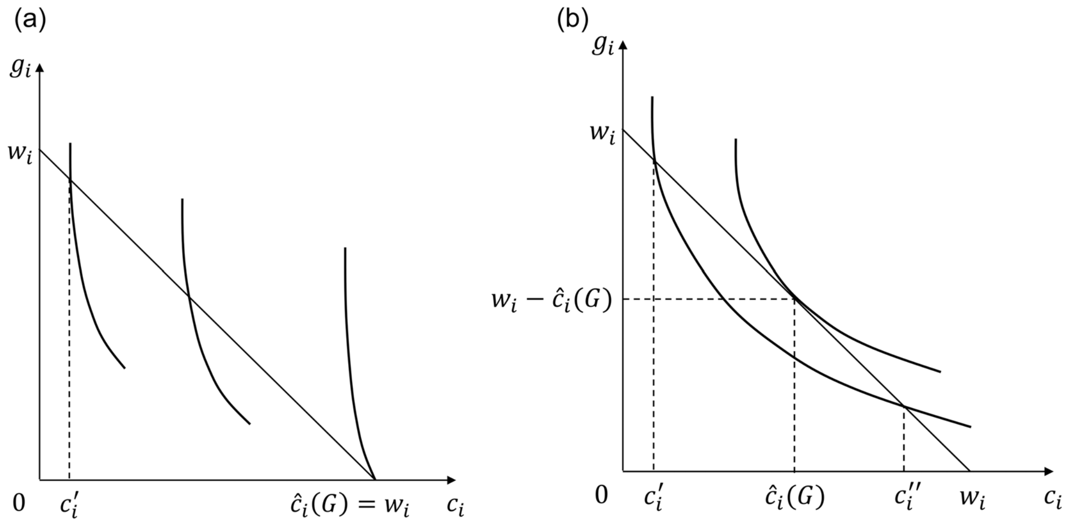

FIGURE A2 (a) Illustration of the proof of Lemma 1 (i). (b) Illustration of the proof of Lemma 1 (ii).

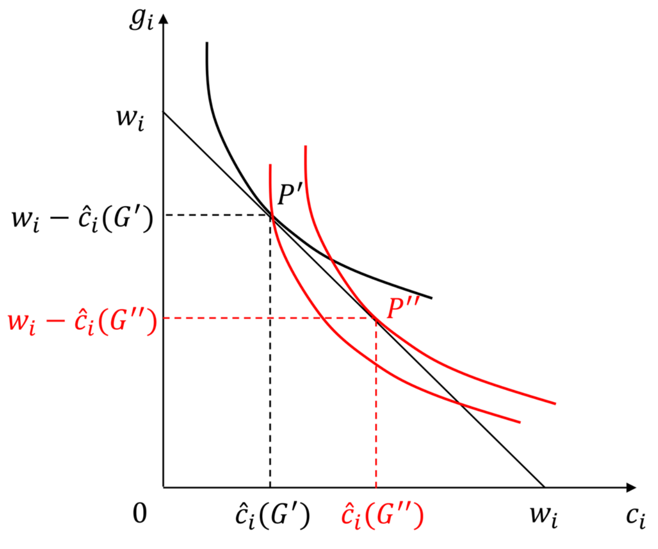

FIGURE A3 Illustration of the proof of Lemma 2

existence of $\tilde{c}_i(G)$ follows from the intermediate value theorem. Uniqueness is implied by $m_1^i(c_i, G) < 0$ for $c_i < \hat{c}_i(G)$.

(ii) $\tilde{c}_i(G) < \hat{c}_i(G)$ follows from $1 = m^i(\tilde{c}_i(G), G) > m^i(\hat{c}_i(G), \hat{G}) = 0$ and $m_1^i(c_i, G) < 0$. By differentiating $m^i(\tilde{c}_i(G), G) = 1$ with respect to G we obtain $\frac{\partial \tilde{c}_i}{\partial G} = -\frac{m_2^i(\tilde{c}_i(G), G)}{m_1^i(\tilde{c}_i(G), G)} > 0$, which follows from (A3), that is, from $m_1^i(c_i, G) < 0$ and $m_2^i(c_i, G) > 0$. A visualization of the proof for $\frac{\partial \tilde{c}_i}{\partial G} > 0$ is given in Figure A1 above: Assume that $c_i' = \tilde{c}_i(G')$ holds at $P' = (c_i', G')$ and that public good supply is increased to G'' on a higher indifference curve. The condition $m_2^i(c_i, G) > 0$ as part of (A3) then implies $m^i(c_i', G''') > m^i(c_i', G') = 1$ while $m_1^i(c_i, G) < 0$

implies $m^i(c_i''', G') < m^i(c_i', G') = 1$. To achieve $m^i(c_i'', G'') = 1$ and thus $\tilde{c}_i(G'') = c_i''$ the point P'' must lie between P_1' and P_2' on the common indifference curve, which gives $c_i'' > c_i'$ as depicted in Figure A1. □

Proof of Proposition 1. (i) Since $m^i(w_i, \hat{G}_i) = 0$, $\lim_{G\to\infty} m^i(w_i, G) = \infty$ and $m_2^i(w_i, G) > 0$ by (A3), the intermediate value theorem gives the existence of a unique $\breve{G}_i$ with $\breve{G}_i > \hat{G}_i$ where $m^i(w_i, \breve{G}_i) = 1$ holds.

(ii) As by Lemma 3 $\tilde{c}_i(G)$ is increasing and $\tilde{c}_i(G) < \hat{c}_i(G)$ we have $\tilde{c}_i = \lim_{G\to\infty} \tilde{c}_i(G) \le \hat{c}_i$. Now assume $\tilde{c}_i < \hat{c}_i$. But as a consequence of (A3) there exists some G' where $m^i(\tilde{c}_i, G') > 1$. Due to $m_1^i(c_i, G) < 0$ we then obtain $\tilde{c}_i(G') > \tilde{c}_i$. This is a contradiction. □

Proof of Proposition 3. For PPG utility functions in general and thus for the auxiliary utility function in particular we have $m_1^i < 0$ if and only if $u_{11}^i - m^i u_{12}^i < 0$ and $m_2^i > 0$ if and only if $u_{12}^i - m^i u_{22}^i > 0$, where u_{11}^i, u_{22}^i and u_{12}^i are the second partial derivatives of the utility function $u^i(c_i, G)$. (This follows directly by taking the derivative of $m^i(c_i, G) = \frac{u_1^i(c_i, G)}{u_2^i(c_i, G)}$ with respect to c_i and G, respectively, which gives $m_1^i = \frac{u_{11}^i - m^i u_{12}^i}{u_2^i}$ and $m_2^i = \frac{u_{12}^i - m^i u_{22}^i}{u_2^i}$). Turning to the originally given IPG function $U^i(c_i, G, g_i)$ and observing that $u_{11}^i = U_{11}^i + U_{33}^i - 2U_{13}^i$, $u_{12}^i = u_{21}^i = U_{12}^i - U_{23}^i$ and $u_{22}^i = U_{22}^i$, the conditions for $m_1^i < 0$ and $m_2^i > 0$ are fulfilled for the auxiliary PPG utility function $u^i(c_i, G) = U^i(c_i, G, w_i - c_i)$ under the assumptions of Proposition 3. □

Proof of Proposition 5. According to Proposition 2 for any number k of agents public good supply $G^N(k)$ is characterized by the condition

$$G^N(k) + k\tilde{c}(G^N(k)) = kw. \tag{4}$$

Now assume that $G^N(k)$ is bounded from above by some $\bar{G}$ for all $k \in \mathbb{N}$. Since $\tilde{c}(G)$ is increasing in G, we get $\tilde{c}(G(k)) < \tilde{c}(\bar{G}) < w$ and, consequently, $g^N(k) = w - \tilde{c}(G(k)) > w - \tilde{c}(\bar{G}) > 0$ for all k. As k can be chosen so large that $k(w - \tilde{c}(\bar{G})) > \bar{G}$ holds, this gives a contradiction. The rest of the assertion follows from Proposition 1 (ii), that is, from $\lim_{k\to\infty} \tilde{c}(G^N(k)) = \lim_{G\to\infty} \tilde{c}(G) = \hat{c}$. □

Proof of Proposition 8. The change of agent j's marginal willingness to pay that is implied by an increase of w_j is specified by taking the derivative of $m^j(c_j, G) = \frac{U_1^j(c_j, G, w_j - c_j) - U_3^j(c_j, G, w_j - c_j)}{U_2^j(c_j, G, w_j - c_j)}$ with respect to w_j at any point (c_j, G) with $c_j < \hat{c}_j(G)$. This yields

$$\frac{\partial m^j}{\partial w_j} = \frac{(U_{13}^j - U_{33}^j)U_2^j - (U_1^j - U_3^j)U_{23}^j}{(U_2^j)^2}. \tag{5}$$

From (A4) it then directly follows from Equation (5) that $\frac{\partial m^j}{\partial w_j} > 0$. □

Proof of Proposition 9. Let G^N denote public good supply in the Nash equilibrium before the income increase. Making agent j's expansion path also dependent on her income w_j, that is, writing $\tilde{c}_j(G, w_j)$, gives

$$m^j(c_j(G^N, w_j), G^N) = \frac{U_1^j(c_j(G^N, w_j), G^N, w_j - c_j(G^N, w_j)) - U_3^j(c_j(G^N, w_j), G^N, w_j - c_j(G^N, w_j))}{U_2^j(c_j(G^N, w_j), G^N, w_j - c_j(G^N, w_j))} = 1. \tag{6}$$

Taking the derivative of this expression with respect to w_j yields

$$\frac{d\tilde{c}_j}{dw_j} = \frac{U_{13} - U_{23} - U_{33}}{(U_{13} - U_{23} - U_{33}) + (U_{13} - U_{11} + U_{12})}. \tag{7}$$

Given (A4) it follows that $0 < \frac{d\tilde{c}_j}{dw_j} < 1$, which means that at G^N agent j's income expansion path shifts to the right when w_j is increased, but by less than the increase of w_j. The consequence for condition (3) taken at G^N is that the right-hand side of (3) becomes larger than the left-hand side. To restore equilibrium, that is, to attain again equality of both sides of (3), public good supply must increase. All agents $i \neq j$ move outwards their invariant expansion paths, thus increasing their private consumption. The change of agent j's private consumption can be divided into two steps: A move outward her original expansion path, and then a move to the right to the new expansion path that corresponds to her increased endowment.

It is obvious that utility of all agents $i \neq j$ increases. The change of agent j's utility in the Nash equilibrium that is caused by a marginal increase of w_j is $\frac{dU^j}{dw_j} = U_1^j \frac{d\tilde{c}_j}{dw_j} + U_2^j \frac{dG}{dw_j} + U_3^j\left(1 - \frac{d\tilde{c}_j}{dw_j}\right) = U_2^j\left(\frac{d\tilde{c}_j}{dw_j} + \frac{dG^N}{dw_j}\right) + U_3^j > 0$ since along the (original) expansion path $U_1^j - U_3^j = U_2^j$ holds. □

Proof of Proposition 11. (i) We show that a marginal increase of α_j shifts agent j's expansion path to the right. To that end we determine the changes of the marginal rate of substitution $m^j(c_j, G) = \frac{U_1^j(c_j, \alpha_j G, g_j) - U_3^j(c_j, \alpha_j G, g_j)}{\alpha_j U_2^j(c_j, \alpha_j G, g_j)}$ at some point (c_j, G) with $c_j < \hat{c}_j(G)$ when α_j changes. Simplifying the exposition by assuming $\beta_j = 1$ and taking $\alpha_j = 1$ as the starting point gives

$$\frac{\partial m^j}{\partial \alpha_j} = \frac{(U_{12}^j - U_{23}^j)U_2^j G - (U_1^j - U_3^j)(U_{22}^j G + U_2^j)}{(U_2^j)^2} > 0 \tag{8}$$

if assumption (A4) is satisfied and, in addition, $U_{22}^j G + U_2^j < 0$, that is, $-\frac{U_{22}^j G}{U_2^j} > 1$, holds. The assertion then follows from Proposition 6.

(ii) The proof for a change of β_j runs completely analogously. □